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SUMMABILITY OF FOURIER ORTHOGONAL SERIES FOR JACOBI WEIGHT ON A BALL IN $\mathbb{R}^d$

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Abstract. Fourier orthogonal series with respect to the weight function $(1-|\mathbf{x}|^2)^{\mu-1/2}$ on the unit ball in $\mathbb{R}^d$ are studied. Compact formulae for the sum of the product of orthonormal polynomials in several variables and for the reproducing kernel are derived and used to study the summability of the Fourier orthogonal series. The main result states that the expansion of a continuous function in the Fourier orthogonal series with respect to $(1-|\mathbf{x}|^2)^{\mu-1/2}$ is uniformly (C,δ) summable on the ball if and only if $\delta > \mu + (d-1)/2$.

1. Introduction

The purpose of this paper is to study the summability of the Fourier orthogonal series with respect to the weight function $(1-|\mathbf{x}|^2)^{\mu-1/2}$ on the unit ball in $\mathbb{R}^d$. The main result states that the expansion of a continuous function in orthogonal series is uniformly (C,δ) summable if and only if $\delta > \mu + (d-1)/2$, which provides a complete answer for the Cesáro summability for these weight functions. Primitive study in this direction has been conducted for years (cf. [6, Vol. II, Chapter XII]), but the sharp result as such seems to be obtained for the first time. To motivate our approach, we start with some general background on orthogonal polynomials in several variables.

Let Π^d be the space of polynomials in d variables and Π_n^d be the subspace of polynomials of degree at most n. Let W be a nonnegative weight function on $\mathbb{R}^d$ with integral 1. For each $n \in \mathbb{N}_0$, there are $r_n^d = \binom{n+d-1}{n}$ many linearly independent polynomials of degree exactly n in d variables that are mutually orthogonal. The number r_n^d is the same as the number of distinct monomials of degree n, or the cardinality of the set $\{\alpha \in \mathbb{N}_0^d : |\alpha|_1 = n\}$, where $|\alpha|_1$ denote the ℓ^1 norm of α. We denote by $\{P_\alpha^n\}$, $\alpha \in \mathbb{N}_0^d$, $|\alpha|_1 = n$ and $0 \le n < \infty$, one family of orthonormal polynomials that forms a basis of Π^d, where the superscript n means that $P_\alpha^n \in \Pi_n^d$. We arrange the polynomials $\{P_\alpha^n\}_{|\alpha|_1=n}$ according to the lexicographical order as $P_{\alpha_1}, \ldots, P_{\alpha_{r_n^d}}$, $\alpha_i \in \mathbb{N}_0^d$. A useful vector notation

$$\mathbb{P}_n = (P_{\alpha_1}^n, P_{\alpha_2}^n, \ldots, P_{\alpha_{r_n^d}}^n)^T, \tag{1.1}$$

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is introduced in [14]. The orthonormal property of P_α^n means that

$$\int_{\mathbb{R}^d} P_\alpha^n(\mathbf{x}) P_\beta^m(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \delta_{n,m}\delta_{\alpha,\beta}, \quad \text{or} \quad \int_{\mathbb{R}^d} \mathbb{P}_n(\mathbf{x}) \mathbb{P}_m^T(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \delta_{n,m} I,$$

where I is the identity matrix of size r_n^d.

One of the essential problems in dealing with orthogonal polynomials in several variables is the non-uniqueness of the orthogonal basis. Let V_n^d denote the subspace spanned by the polynomials $P_{\alpha_j}^n$, $1 \le j \le r_n^d$. Then it is easy to see that for any orthogonal matrix Q_n of size r_n^d the components of the polynomial vector $Q_n \mathbb{P}_n$ form an orthonormal basis of V_n^d. On the other hand, any two orthonormal bases of V_n^d differ by an orthogonal matrix. It turns out that many results concerning orthogonal polynomials in several variables can be stated uniquely in terms of V_n^d rather than in terms of a particular basis of V_n^d. Based on this principal we have used the notation $\mathbb{P}_n$ in [14] and a number of subsequent papers to study orthogonal polynomials in several variables, in which results parallel to the theory of orthogonal polynomials in one variable are established (cf. [15] and the references there). The principle is particularly evident when one deals with the Fourier orthogonal series. Let f be integrable with respect to W. The Fourier orthogonal expansion of f with respect to a sequence of orthonormal polynomials $\{P_\alpha^n\}$ is given by

$$f \sim \sum_{n=0}^{\infty} \sum_{|\alpha|_1 = n} a_\alpha^n(f) P_\alpha^n(f), \qquad \text{where} \qquad a_\alpha^n(f) = \int P_\alpha^n(\mathbf{x}) f(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}. \tag{1.2}$$

The expansion can be viewed in terms of the V_n^d, which becomes clear when we write the expansion using $\mathbb{P}_n$ as follows:

(1.3)

$$f \sim \sum_{n=0}^{\infty} \mathrm{proj}_{V_n} f = \sum_{n=0}^{\infty} \mathbf{a}_n^T(f) \mathbb{P}_n, \qquad \text{where} \qquad \mathbf{a}_n(f) = \int f(\mathbf{x}) \mathbb{P}_n(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}.$$

In fact, the n-th reproducing kernel of the orthonormal polynomials, defined by

$$\mathbf{K}_n(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{n} \sum_{|\alpha|_1 = k} P_\alpha^k(\mathbf{x}) P_\alpha^k(\mathbf{y}) = \sum_{k=0}^{n} \mathbb{P}_k^T(\mathbf{x}) \mathbb{P}_k(\mathbf{y}), \tag{1.4}$$

is easily seen to depend on the V_k^d's rather than a particular basis of V_k^d. The n-th partial sum $S_n f$ of the expansion can be written in terms of $\mathbf{K}_n(\cdot, \cdot)$ as

$$S_n f = \sum_{k=0}^{n} \mathbf{a}_k^T(f) \mathbb{P}_k = \int \mathbf{K}_n(\cdot, \mathbf{y}) f(\mathbf{y}) W(\mathbf{y}) d\mathbf{y}. \tag{1.5}$$

Clearly, a more basic quantity that depends on V_n^d rather than a particular basis of V_n^d is the sum of orthonormal polynomials $[\mathbb{P}_n(\mathbf{x})]^T \mathbb{P}_n(\mathbf{y})$.

For general weight functions, the summability of $S_n f$ to f has been studied in [16]. The special weight function often warrants better results. In order to achieve sharp results for the summability of the Fourier orthogonal series, it is essential to have a compact formula for $\mathbf{K}_n(\mathbf{x}, \mathbf{y})$. In one variable, this is given by the Christoffel-Darboux formula which works for every weight function. In several variables, however, the corresponding Christoffel-Darboux formula is not enough for this purpose; the compact formula has to be derived case by case. So far, little has been done in this direction. In this paper, we will derive such a formula for the weight function $W_\mu(\mathbf{x}) = w_\mu(1-|\mathbf{x}|^2)^{\mu-1/2}$ on the unit ball, where w_μ is a normalization constant. In fact, we will derive a compact formula for $[\mathbb{P}_n(\mathbf{x})]^T\mathbb{P}_n(\mathbf{y})$, which takes the form

$$[\mathbb{P}_n(\mathbf{x})]^T\mathbb{P}_n(\mathbf{y}) = \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}}\int_0^\pi C_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi)$$
$$\times(\sin\psi)^{2\mu-1}d\psi\Big/\int_0^\pi(\sin\psi)^{2\mu-1}d\psi, \qquad \mathbf{x},\mathbf{y}\in B^d,$$

where $C_n^{(\lambda)}$ denotes the classical Gegenbauer polynomial. Clearly, this formula resembles the product formula for the Gegenbauer polynomials, which is the above formula with $d=1$. It is well-known that the product formula has many interesting applications; it yields, in particular, a convolution structure that has been used to deal with the summability of ultraspherical series. For orthogonal polynomials in several variables, product formulae are known only for a few classical weight functions and they often follow from those of one variable; for example, we mention the product Jacobi weight functions and the weight function $(1-|z|^2)^\mu$ on the unit disk of the complex plane ([10]). Part of the reason is that the product formula depends on the choice of the particular orthogonal basis. For the purpose of studying summability of the orthogonal series, however, the compact formula for $[\mathbb{P}_n(\mathbf{x})]^T\mathbb{P}_n(\mathbf{y})$ turns out to be sufficient; it enables us to prove our result on the Cesáro summability, which states that the Fourier orthogonal series with respect to W_μ is uniformly (C,δ) summable if and only if $\delta>\mu+(d-1)/2$. Moreover, the compact formula enables us to extend several inequalities for the sums of ultraspherical polynomials to several variables, including an inequality of Askey and Gasper, from which follows that the (C,δ) means of the Fourier orthogonal series with respect to W_μ is positive if and only if $\delta\ge 2\mu+d$.

The paper is organized as follows. In Section 2, we fix notation and present a family of orthonormal polynomials. The compact formula for $[\mathbb{P}_n(\mathbf{x})]^T\mathbb{P}_n(\mathbf{y})$ is proved in Section 3; the results on positive sums and preliminary on summability are presented in Section 4. The main results on summability are given in Section 5.

2. Definition and preliminary

Throughout this paper we write $\mathbf{x}=(x_1,\dots x_d)^T\in\mathbb{R}^d$ and $\mathbf{x}\cdot\mathbf{y}=x_1y_1+\dots+x_dy_d$ for the standard inner product of $\mathbb{R}^d$. We use the notation $|\cdot|$ for the Euclidean norm $|\mathbf{x}|^2=\mathbf{x}\cdot\mathbf{x}$ and we write $|\mathbf{x}|_1=|x_1|+\dots+|x_d|$. Throughtout this paper we use B^d to denote the unit ball in $\mathbb{R}^d$ and S^{d-1} to denote the unit sphere in $\mathbb{R}^d$; that is,

$$B^d=\{\mathbf{x}\in\mathbb{R}^d:|\mathbf{x}|\le 1\} \qquad \text{and} \qquad S^{d-1}=\{\mathbf{x}\in\mathbb{R}^d:|\mathbf{x}|=1\}.$$

The weight function that we deal with in this paper is the normalized function

$$W_\mu(\mathbf{x})=W_{\mu,d}(\mathbf{x})=w_\mu(1-|\mathbf{x}|^2)^{\mu-\frac12}, \quad \mu\ge 0, \qquad \mathbf{x}\in B^d, \tag{2.1}$$

where w_μ is a constant chosen so that the integral of W_μ is 1,

$$w_\mu=w_{\mu,d}=\frac{2}{\omega_{d-1}}\frac{\Gamma(\mu+\frac{d+1}{2})}{\Gamma(\mu+\frac12)\Gamma(\frac d2)}=\frac{\Gamma(\mu+\frac{d+1}{2})}{\pi^{d/2}\Gamma(\mu+\frac12)}. \tag{2.2}$$

Here and in the following, we use ω_{d-1} to denote the surface area of S^{d-1}; it is known that $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$. The value of w_μ can be verified by the use of the standard coordinates $\mathbf{x} = r\mathbf{x}'$, $\mathbf{x}' \in S^{d-1}$, as follows:

$$\begin{aligned}\int_{B^d}(1-|\mathbf{x}|^2)^{\mu-\frac{1}{2}}d\mathbf{x} &= \int_0^1\int_{S^{d-1}} d\omega(1-r^2)^{\mu-\frac{1}{2}}r^{d-1}dr\\ &= \frac{\omega_{d-1}}{2}\int_0^1(1-t)^{\mu-\frac{1}{2}}t^{\frac{d-2}{2}}dt.\end{aligned}$$

The weight function $W_\mu : \mathbb{R}^d \mapsto \mathbb{R}$ is a radial function; occasionally we will write $W_\mu(r)$ with $r \in \mathbb{R}$, as if it is a function from $\mathbb{R} \mapsto \mathbb{R}$, the slight abuse of notation should not cause any confusion. Whenever it is necessary we will write $W_{\mu,d}$ to emphasis the dependence on d. For $d = 1$, the orthogonal polynomials with respect to the weight function W_μ are the ultraspherical polynomials, customarily denoted by $C_n^{(\mu)}$, which is why we use the power $\mu - (1/2)$ instead of μ in the definition of W_μ.

To describe orthogonal polynomials with respect to W_μ, we will need to recall the definition of ultraspherical polynomials. The basics are contained in [13, p. 80]. We are interested in the polynomials $C_n^{(\lambda)}$ for $\lambda \geq 0$. They are orthogonal with respect to $(1-x^2)^{\lambda-1/2}$ on $[-1, 1]$ and they satisfy

(2.3)
$$\int_{-1}^1\Big[C_n^{(\lambda)}(x)\Big]^2(1-x^2)^{\lambda-\frac{1}{2}}dx = 2^{1-2\lambda}\pi\big[\Gamma(\lambda)\big]^{-2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)\Gamma(n+1)} =: h_n, \quad \lambda > 0,$$

where for $\lambda = 0$, the above relation holds under the limit relation

$$\lim_{\lambda\to 0}\frac{\lambda+n}{\lambda}C_n^{(\lambda)}(\cos\theta) = \begin{cases}1, & \text{for } n = 0,\\ 2\cos n\theta, & \text{for } n = 1, 2, \ldots.\end{cases} \tag{2.4}$$

These polynomials enjoy many properties, some of them are given in the end of the section.

We now present one family of orthonormal polynomials with respect to W_μ explicitly. For $d \geq 2$, we denote these polynomials by $\{P_\alpha^{n,(\mu)}\}$ or $\{P_{\alpha,d}^{n,(\mu)}\}$ when we need to emphasis the dependence on d. For the simplicity of the notation, it is more convenient to deal with the index $\lambda = \mu + (d-1)/2$; *i.e.*, we work with $W_{\lambda-(d-1)/2}$. For $d = 2$, these orthonormal polynomials are given in [11] by

$$P_k^{n,(\lambda-\frac{1}{2})}(x,y) = [h_{k,2}^n]^{-\frac{1}{2}}C_{n-k}^{(\lambda+k)}(x)(1-x^2)^{k/2}C_k^{(\lambda-\frac{1}{2})}(y(1-x^2)^{-\frac{1}{2}}), \quad 0 \leq k \leq n,$$

where $h_{k,2}^n$ are constants chosen so that $P_k^{n,(\mu)}$ are normalized; in this case, it is more natural to use the subscript k instead of $\alpha = (k, n-k)$. In general, for $d \geq 2$, we need the following notation. For $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$, we denote by $\mathbf{x}_j \in \mathbb{R}^j$,

$$\mathbf{x}_0 = 0, \qquad \mathbf{x}_j = (x_1, \ldots, x_j)^T, \qquad 1 \leq j \leq d-1. \tag{2.5}$$

For $\alpha \in \mathbb{N}_0$, $|\alpha|_1 = n$, we rewrite it as

$$\alpha = (k_1, \ldots, k_{d-1}, n - k_1 - \ldots - k_{d-1}) = (k_1, \ldots, k_{d-1}, n - |\mathbf{k}|_1),$$

where $|\mathbf{k}|_1 = k_1 + \ldots + k_{d-1}$. Let $n \in \mathbb{N}_0$ and $n = k_0 \geq k_1 \geq k_2 \geq \ldots \geq k_{d-1} \geq k_d = 0$. For $\alpha = (k_1, \ldots, k_{d-1}, n - |\mathbf{k}|_1)$, we define the polynomial $P_\alpha^{n,(\lambda-\frac{d-1}{2})}$,

$\lambda > (d-1)/2$, by

$$(2.6)\quad P_\alpha^{n,(\lambda-\frac{d-1}{2})}(\mathbf{x}) = [h_{\alpha,d}^n]^{-\frac{1}{2}} \prod_{j=0}^{d-1} (1-|\mathbf{x}_j|^2)^{\frac{k_j-k_{j+1}}{2}} C_{k_j-k_{j+1}}^{(\lambda+k_{j+1}-\frac{j}{2})}\Big(\frac{x_{j+1}}{\sqrt{1-|\mathbf{x}_j|^2}}\Big),$$

where

(2.7)

$$h_{\alpha,d}^n = \frac{\Gamma(\lambda+1)\pi^{\frac{d}{2}}}{\Gamma(\lambda+1-\frac{d}{2})} \prod_{j=1}^{d} \frac{2^{j+1-2\lambda-2k_j}\Gamma(2\lambda+k_{j-1}+k_j-j+1)}{(2\lambda+2k_{j-1}-j+1)\Gamma(k_{j-1}-k_j+1)[\Gamma(\lambda+k_j-\frac{j-1}{2})]^2}.$$

Moreover, for $\lambda = (d-1)/2$, we also define $P_\alpha^{n,(0)}$ by the formula (2.6) with the understanding that the limit (2.4) is used whenever $\lambda \to 0$; the formula (2.7) for $h_{\alpha,d}^n$ in the case of $\lambda = (d-1)/2$ has an additional factor $\eta_{k_{d-1}}$, which takes the value $1/2$ if $k_{d-1} > 0$ and 1 if $k_{d-1} = 0$.

Proposition 2.1. *The family of polynomials $\{P_\alpha^{n,(\mu)}\}$ is orthonormal with respect to W_μ on B^d; i.e.,*

$$\int_{B^d} P_\alpha^{n,(\mu)}(\mathbf{x}) P_\beta^{m,(\mu)}(\mathbf{x}) W_\mu(\mathbf{x}) d\mathbf{x} = \delta_{\alpha,\beta}\delta_{n,m}.$$

Proof. Again we work with $\mu = \lambda - (d-1)/2$. Throughout this proof we write, see (2.6),

$$\tilde{P}_{\alpha,d}^{n,(\lambda-\frac{d-1}{2})}(\mathbf{x}) = \prod_{j=0}^{d-1} (1-|\mathbf{x}_j|^2)^{\frac{k_j-k_{j+1}}{2}} C_{k_j-k_{j+1}}^{(\lambda+k_{j+1}-\frac{j}{2})}\Big(\frac{x_{j+1}}{\sqrt{1-|\mathbf{x}_j|^2}}\Big).$$

Let $\alpha' = (k_1-k_{d-1}, \ldots, k_{d-2}-k_{d-1}, n-|\mathbf{k}|_1+(d-2)k_{d-1})$. Then it follows from the definition that

$$\tilde{P}_{\alpha,d}^{n,(\lambda-\frac{d-1}{2})}(\mathbf{x}) = \tilde{P}_{\alpha',d-1}^{n-k_{d-1},(\lambda+k_{d-1}-\frac{d-2}{2})}(\mathbf{x}_{d-1}) \times (1-|\mathbf{x}_{d-1}|^2)^{\frac{k_{d-1}}{2}} C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}\Big(\frac{x_d}{\sqrt{1-|\mathbf{x}_{d-1}|^2}}\Big).$$

If we write $\beta = (j_1, \ldots, j_{d-1}, m-|\mathbf{j}|)$, then it follows that

$$\begin{aligned}
&\int_{B^d} \tilde{P}_\alpha^{n,(\lambda-\frac{d-1}{2})}(\mathbf{x}) \tilde{P}_\beta^{m,(\lambda-\frac{d-1}{2})}(\mathbf{x})(1-|\mathbf{x}|^2)^{\lambda-\frac{d}{2}} d\mathbf{x} \\
&= \int_{B^{d-1}} \tilde{P}_{\alpha',d-1}^{n-k_{d-1},(\lambda+k_{d-1}-\frac{d-2}{2})}(\mathbf{x}_{d-1}) \tilde{P}_{\beta',d-1}^{m-j_{d-1},(\lambda+j_{d-1}-\frac{d-2}{2})}(\mathbf{x}_{d-1}) \\
&\quad \times (1-|\mathbf{x}_{d-1}|^2)^{\lambda+\frac{k_{d-1}+j_{d-1}-d+1}{2}} d\mathbf{x}_{d-1} \int_{-\sqrt{1-|\mathbf{x}_{d-1}|^2}}^{\sqrt{1-|\mathbf{x}_{d-1}|^2}} C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}\Big(\frac{x_d}{\sqrt{1-|\mathbf{x}_{d-1}|^2}}\Big) \\
&\quad \times C_{j_{d-1}}^{(\lambda-\frac{d-1}{2})}\Big(\frac{x_d}{\sqrt{1-|\mathbf{x}_{d-1}|^2}}\Big) \times \Big(1-\frac{x_d^2}{1-|\mathbf{x}_{d-1}|^2}\Big)^{\lambda-\frac{d}{2}} \frac{dx_d}{\sqrt{1-|\mathbf{x}_{d-1}|^2}}.
\end{aligned}$$

Changing variable $x_d \mapsto \sqrt{1-|\mathbf{x}_{d-1}|^2}\, t$, the second integral is seen to be equal to

$$\delta_{k_{d-1},j_{d-1}} \int_{-1}^{1} \Big[C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}(t)\Big]^2 (1-t^2)^{\lambda-\frac{d}{2}} dt.$$

The first integral with $k_{d-1} = j_{d-1}$ allows us to continue this process. To the end, we conclude that the desired equation holds true with $h^n_{\alpha,d}$ given by

$$h^n_{\alpha,d} = w_{\lambda-\frac{d-1}{2},d} \prod_{j=1}^{d} \int_{-1}^{1} \left[C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(t) \right]^2 (1-t^2)^{\lambda+k_j-\frac{j}{2}} dt.$$

The formula for $h^n_{\alpha,d}$ follows from inserting (2.2) and (2.3) to the above equation. $\square$

It should be pointed out that the study of orthogonal polynomials with respect to $W_{\mu,d}$ has been undertaken for many years (cf. [6, Vol. II, Chapter XII]). In the early studies, instead of using orthonormal polynomials, the polynomials $V_\alpha^{n,(\mu)} \in \Pi_n^d$, $|\alpha|_1 = n$, defined by

$$(1 - 2\mathbf{a}\cdot\mathbf{x} + |\mathbf{a}|^2)^{-\mu-\frac{d-1}{2}} = \sum_{\alpha\in\mathbb{N}_0^d} \mathbf{a}^\alpha V_\alpha^{n,(\mu)}(\mathbf{x}), \qquad \mathbf{a},\mathbf{x} \in \mathbb{R}^d, \tag{2.8}$$

play an essential role, which can be traced all the way back to the work of Hermite. The polynomials $V_\alpha^{n,(\mu)}$ and $V_\beta^{m,(\mu)}$ are orthogonal with respect to $W_{\mu,d}$ if $m \ne n$, but they are not orthogonal if $m = n$ and the components of $\alpha-\beta$ are even integers. Therefore, $\{V_\alpha^{n,(\mu)}\}_{|\alpha|_1=n}$ is a basis for the subspace $\mathcal{V}_n^d$ introduced in Section 1, but the basis is not an orthogonal one. It implies, in particular, that $P_\alpha^{n,(\mu)}$ can be written as a linear combination of $V_\alpha^{n,(\mu)}$. It follows from [6, Vol. II, p. 275, (14)] that for each fixed n the polynomials $P_\alpha^{n,(\mu)}$ satisfy the partial differential equation

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j} \left\{ \frac{\partial F}{\partial x_j} - x_j \left[(2\mu-1)F + \sum_{k=1}^{d} x_k \frac{\partial F}{\partial x_k} \right] \right\} = -(n+d)(n+2\mu-1)F.$$

In other words, $P_\alpha^{n,(\mu)}$ are the eigenfunctions of a differential operator.

The ultraspherical polynomials enjoy many properties, several of them that we shall need are recorded below. Two basic ones are [13, p. 80, (4.7.3) and (4.7.4)]

$$C_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n}, \qquad C_n^{(\lambda)}(-x) = (-1)^n C_n^{(\lambda)}(x). \tag{2.9}$$

Let us denote by $\tilde{C}_n^{(\lambda)}$ the orthonormal polynomial of degree n with respect to the normalized weight function $W_{\lambda,1}$. Then, by (2.3), it follows that $\tilde{C}_n^{(\lambda)} = h_n^{-1/2} C_n^{(\lambda)}$ and

$$\tilde{C}_n^{(\lambda)}(1)\tilde{C}_n^{(\lambda)}(x) = \frac{n+\lambda}{\lambda} C_n^{(\lambda)}(x). \tag{2.10}$$

One deep property we shall need is the addition formula of Gegenbauer, which states that [6, Vol. I, Sec. 3.15.1, (19)]

$$\begin{aligned} C_n^{(\lambda)}&(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi) \\ &= \sum_{k=0}^{n} a_{k,n}^\lambda (\sin\theta)^k C_{n-k}^{(\lambda+k)}(\cos\theta)(\sin\phi)^k C_{n-k}^{(\lambda+k)}(\cos\phi) C_k^{(\lambda-\frac12)}(\cos\psi), \end{aligned} \tag{2.11}$$

with

$$a_{k,n}^\lambda = \frac{\Gamma(2\lambda-1)2^{2k}\big[\Gamma(k+\lambda)\big]^2 (n-k)!(2k+2\lambda-1)}{\big[\Gamma(\lambda)\big]^2 \Gamma(n+k+2\lambda)}. \tag{2.12}$$

When $\lambda = 1/2$, we factor out $(2k+\lambda-1)$ from $a^{\lambda}_{k,n}$ in (2.12) and use the limit (2.4) in the formula (2.11). For $\psi = 0$ and $\lambda \to 0$, the addition formula (2.11) is reduced to the addition formula for cosine polynomials. From (2.12) and (2.9) it is easy to verify that $a^{\lambda}_{0,n} = [C_n^{(\lambda)}(1)]^{-1}$. Thus, from (2.11) follows the Gegenbauer's product formula [6, Vol. I, Sec. 3.15.1, (20)]

(2.13)
$$\frac{C_n^{(\lambda)}(\cos\theta)C_n^{(\lambda)}(\cos\phi)}{C_n^{(\lambda)}(1)} = c_\lambda \int_0^\pi C_n^{(\lambda)}(\cos\theta\cos\phi + \sin\theta\sin\phi\cos\psi)(\sin\psi)^{2\lambda-1}d\psi,$$

where $\lambda > 0$ and

$$c_\lambda^{-1} = \int_0^\pi (\sin\psi)^{2\lambda-1}d\psi = \int_{-1}^1 (1-t^2)^{\lambda-1}dt = w^{-1}_{\lambda-\frac{1}{2},1}. \tag{2.14}$$

The ultraspherical polynomials are special cases of the Jacobi polynomials. The latter polynomials, usually denoted by $P_n^{(\alpha,\beta)}$, are orthogonal with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, $\alpha,\beta > -1$. We use the standard notation as in [13, p. 58]. The relation between the ultraspherical polynomials and the Jacobi polynomials is given by [13, p. 80, (4.7.1)]

$$C_n^{(\lambda)}(x) = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda)}{\Gamma(2\lambda)\Gamma(n+\lambda+\frac{1}{2})} P_n^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x). \tag{2.15}$$

Other properties of these polynomials that are needed will be given in the place where they are used.

3. Compact formulae

The main results in this section are the compact formulae for $\left[\mathbb{P}_n^{(\mu)}(\mathbf{x})\right]^T\mathbb{P}_n^{(\mu)}(\mathbf{y})$ and $\mathbf{K}_n^{(\mu)}(\mathbf{x},\mathbf{y})$, which are independent of the choice of the orthonormal bases.

Theorem 3.1. *For W_μ on B^d, $\mu > 0$,*

$$\begin{aligned} \left[\mathbb{P}_n^{(\mu)}(\mathbf{x})\right]^T\mathbb{P}_n^{(\mu)}(\mathbf{y}) = & \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} \\ & \times \int_0^\pi C_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi) \\ & \times (\sin\psi)^{2\mu-1}d\psi \Big/ \int_0^\pi (\sin\psi)^{2\mu-1}d\psi, \qquad \mathbf{x},\mathbf{y} \in B^d, \end{aligned} \tag{3.1}$$

and, for $\mu = 0$,

$$\begin{aligned} \left[\mathbb{P}_n^{(0)}(\mathbf{x})\right]^T\mathbb{P}_n^{(0)}(\mathbf{y}) = \frac{n+\frac{d-1}{2}}{d-1} \Big[& C_n^{(\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \\ & + C_n^{(\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})\Big], \qquad \mathbf{x},\mathbf{y} \in B^d. \end{aligned} \tag{3.2}$$

Proof. For $\mathbf{x} \in B^d$ we define a mapping $\mathbf{x} \mapsto (\theta_1, \ldots, \theta_d)$ by

$$x_1 = \cos\theta_1, \quad \frac{x_2}{\sqrt{1-x_1^2}} = \cos\theta_2, \quad \frac{x_d}{\sqrt{1-x_1^2-\ldots-x_{d-1}^2}} = \cos\theta_d.$$

Moreover, for $\mathbf{y} \in B^d$ we associate it with $(\phi_1, \ldots, \phi_d)$ through this mapping. Using the notation $\mathbf{x}_j$ in (2.5), we write

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} = \mathbf{x}_{d-1}\cdot\mathbf{y}_{d-1} + x_d y_d = \mathbf{x}_{d-1}\cdot\mathbf{y}_{d-1} \\ + \sqrt{1-|\mathbf{x}_{d-1}|^2}\sqrt{1-|\mathbf{y}_{d-1}|^2}\,\cos\theta_d\cos\phi_d,\end{aligned}$$

from which and the fact that $\sqrt{1-|\mathbf{x}|^2} = \sqrt{1-|\mathbf{x}_{d-1}|^2}\,\sin\theta_d$ it follows readily that

$$\begin{aligned}&\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\,\cos\psi \\ &\quad = \mathbf{x}_{d-1}\cdot\mathbf{y}_{d-1} + \sqrt{1-|\mathbf{x}_{d-1}|^2}\sqrt{1-|\mathbf{y}_{d-1}|^2}\,(\cos\theta_d\cos\phi_d + \sin\theta_d\sin\phi_d\cos\psi).\end{aligned}$$

We write $\psi_d = \psi$ and define ψ_j, $1 \le j \le d-1$, by

$$\cos\psi_j = \cos\theta_{j+1}\cos\phi_{j+1} + \sin\theta_{j+1}\sin\phi_{j+1}\cos\psi_{j+1}. \tag{3.3}$$

Then we can continue the above process inductively to conclude that

$$\begin{aligned}\mathbf{x}\cdot\mathbf{y} &+ \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\,\cos\psi \\ &= \mathbf{x}_{d-1}\cdot\mathbf{y}_{d-1} + \sqrt{1-|\mathbf{x}_{d-1}|^2}\sqrt{1-|\mathbf{y}_{d-1}|^2}\,\cos\psi_{d-1} \\ &= \ldots = \mathbf{x}_1\cdot\mathbf{y}_1 + \sqrt{1-|\mathbf{x}_1|^2}\sqrt{1-|\mathbf{y}_1|^2}\,\cos\psi_1.\end{aligned}$$

Since $\mathbf{x}_1 = x_1 = \cos\theta_1$ and $\mathbf{y}_1 = y_1 = \cos\phi_1$, we can use the addition formula (2.11) for the ultraspherical polynomials to conclude that

$$\begin{aligned}&C_n^{(\lambda)}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\,\cos\psi) \\ &\quad = C_n^{(\lambda)}(\mathbf{x}_1\cdot\mathbf{y}_1 + \sqrt{1-|\mathbf{x}_1|^2}\sqrt{1-|\mathbf{y}_1|^2}\,\cos\psi_1) \\ &\quad = \sum_{k_1=0}^{n} a_{k_1,n}^{\lambda}(\sin\theta_1)^{k_1} C_{n-k_1}^{(\lambda+k_1)}(\cos\theta_1)(\sin\phi_1)^{k_1} C_{n-k_1}^{(\lambda+k_1)}(\cos\phi_1) C_{k_1}^{(\lambda-\frac{1}{2})}(\cos\psi_1).\end{aligned}$$

Formula (3.3) for ψ_1 allows us to use the addition formula on $C_{k_1}^{(\lambda-\frac{1}{2})}(\cos\psi_1)$; the new formula so derived contains $C_{k_2}^{(\lambda-1)}(\cos\psi_2)$, which allows us to continue the process inductively. Consequently, we conclude that for $\lambda > (d-1)/2$,

$$\begin{aligned}&C_n^{(\lambda)}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\,\cos\psi) \\ &\quad = \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}} a_{k_1,n}^{\lambda} a_{k_1,k_2}^{\lambda-\frac{1}{2}}\cdots a_{k_{d-1},k_{d-2}}^{\lambda-\frac{d-2}{2}} \\ &\qquad \times (\sin\theta_1)^{k_1} C_{n-k_1}^{(\lambda+k_1)}(\cos\theta_1)(\sin\phi_1)^{k_1} C_{n-k_1}^{(\lambda+k_1)}(\cos\phi_1)\cdots(\sin\theta_{d-1})^{k_{d-1}} \\ &\qquad \times C_{k_{d-2}-k_{d-1}}^{(\lambda-\frac{d-2}{2}+k_{d-1})}(\cos\theta_{d-1})(\sin\phi_{d-1})^{k_{d-1}} C_{k_{d-2}-k_{d-1}}^{(\lambda-\frac{d-2}{2}+k_{d-1})} \\ &\qquad \times (\cos\phi_{d-1}) C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}(\cos\psi_{d-1}) \\ &\quad = \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}}\prod_{j=1}^{d-1} a_{k_j,k_{j-1}}^{\lambda-\frac{j-1}{2}}(\sin\theta_j)^{k_j} C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\theta_j)(\sin\phi_j)^{k_j} \\ &\qquad \times C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\phi_j) C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}(\cos\psi_{d-1}).\end{aligned} \tag{3.4}$$

We integrate the above equation with respect to $(\sin\psi)^{2\lambda-d}d\psi$, using (3.3), and apply the product formula (2.13) to conclude that

$$
\begin{aligned}
(3.5)\qquad & c_{\lambda-\frac{d-1}{2}}\int_0^\pi C_n^{(\lambda)}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\,\cos\psi)(\sin\psi)^{2\lambda-d}d\psi \\
&= \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}}\prod_{j=1}^{d} a_{k_j,k_{j-1}}^{\lambda-\frac{j-1}{2}}(\sin\theta_j)^{k_j}C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\theta_j)\\
&\quad\times(\sin\phi_j)^{k_j}C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\phi_j),
\end{aligned}
$$

where we have used the fact that $k_d = 0$ and

$$a_{k_d,k_{d-1}}^{\lambda-\frac{d-1}{2}} = a_{0,k_{d-1}}^{\lambda-\frac{d-1}{2}} = \left[C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}(1)\right]^{-1}.$$

On the other hand, from the definition of our orthonormal polynomials in (2.6), we have that

$$
\begin{aligned}
(3.6)\qquad & \left[\mathbb{P}_n^{(\lambda-\frac{d-1}{2})}(\mathbf{x})\right]^T\mathbb{P}_n^{(\lambda-\frac{d-1}{2})}(\mathbf{y}) = \sum_{|\alpha|_1=n}P_\alpha^n(\mathbf{x})P_\alpha^n(\mathbf{y})\\
&= \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}}\left[h_{\alpha,d}^n\right]^{-1}\prod_{j=1}^{d}(\sin\theta_j)^{k_j}C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\theta_j)\\
&\quad\times(\sin\phi_j)^{k_j}C_{k_{j-1}-k_j}^{(\lambda+k_j-\frac{j-1}{2})}(\cos\phi_j).
\end{aligned}
$$

Moreover, from the formulae (2.7) and (2.12) it follows readily that

$$\left[h_{\alpha,d}^n\right]^{-1} = \frac{\pi^{\frac{d}{2}}\Gamma(\lambda+1-\frac{d}{2})}{\Gamma(\lambda+1)}\frac{2\lambda+2n}{2\lambda-d}\prod_{j=1}^{d}\frac{2^{2\lambda-j-1}}{\pi}\frac{\left[\Gamma(\lambda-\frac{j-1}{2})\right]^2}{\Gamma(2\lambda-j)}\prod_{j=1}^{d}a_{k_j,k_{j-1}}^{\lambda-\frac{j-1}{2}},$$

which can be simplified by the use of the identity (cf. [1, p. 256, (6.1.18)])

$$\Gamma(2\lambda-j) = \frac{2^{2\lambda-j-1}}{\sqrt{\pi}}\Gamma(\lambda-\frac{j-1}{2})\Gamma(\lambda-\frac{j}{2});$$

the result is

$$\left[h_{\alpha,d}^n\right]^{-1} = \frac{\lambda+n}{\lambda}\prod_{j=1}^{d}a_{k_j,k_{j-1}}^{\lambda-\frac{j-1}{2}}.$$

Using this identity and comparing (3.5) and (3.6), we obtain the desired formula (3.1) for $\mu = \lambda-(d-1)/2 > 0$.

To prove (3.2) in the case $\mu = 0$, or $\lambda = (d-1)/2$, we use the limit relation (2.4). We note that it follows from (3.3) and the elementary trigonometric identities such that

$$\cos\psi_{d-1} = \cos(\theta_d-\phi_d),\quad \text{if}\quad \psi = 0;\qquad \cos\psi_{d-1} = \cos(\theta_d+\phi_d),\quad \text{if}\quad \psi=\pi.$$

These formulae are used to derive from (3.4), taking into account (2.4), that

$$
\begin{aligned}
&\frac{n+\frac{d-1}{2}}{d-1}C_n^{(\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y}\pm\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) = \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}}\prod_{j=1}^{d-1}a_{k_j,k_{j-1}}^{\frac{d-j}{2}}(\sin\theta_j)^{k_j}\\
&\quad\times C_{k_{j-1}-k_j}^{(k_j+\frac{d-j}{2})}(\cos\theta_j)(\sin\phi_j)^{k_j}C_{k_{j-1}-k_j}^{(k_j+\frac{d-j}{2})}(\cos\phi_j)C_{k_{d-1}}^{(\lambda-\frac{d-1}{2})}(\cos(\theta_d\mp\phi_d)).
\end{aligned}
$$

On the other hand, from the addition formula for cosine, we have

$$C_m^{(0)}(\cos\theta_d)C_m^{(0)}(\cos\phi_d) = C_m^{(0)}(\cos(\theta_d-\phi_d)) + C_m^{(0)}(\cos(\theta_d+\phi_d))$$

for $m>0$, while for $\mu=0$ the right hand side has a factor $1/2$. Thus, it follows as in (3.6) that

$$\begin{aligned}\left[\mathbb{P}_n^{(0)}(\mathbf{x})\right]^T\mathbb{P}_n^{(0)}(\mathbf{y}) &= \sum_{k_1=0}^{n}\cdots\sum_{k_{d-1}=0}^{k_{d-2}}\left[h_{\alpha,d}^n\right]^{-1}\prod_{j=1}^{d-1}(\sin\theta_j)^{k_j}C_{k_{j-1}-k_j}^{(k_j+\frac{d-j}{2})}(\cos\theta_j)\\ &\quad\times(\sin\phi_j)^{k_j}C_{k_{j-1}-k_j}^{(k_j+\frac{d-j}{2})}(\cos\phi_j)\left[C_{k_{d-1}}^{(0)}(\cos(\theta_d-\phi_d))+C_{k_{d-1}}^{(0)}(\cos(\theta_d+\phi_d))\right].\end{aligned}$$

To conclude the proof, we compare the coefficients of the two expressions. We omit the details. □

It is worthwhile to mention that if we restrict $\mathbf{y}$ to the surface of the ball by setting $|\mathbf{y}|=1$, then the compact formula (3.1) or (3.2) takes the form

$$\left[\mathbb{P}_n^{(\mu)}(\mathbf{x})\right]^T\mathbb{P}_n^{(\mu)}(\mathbf{y}) = \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}}C_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y}), \qquad |\mathbf{y}|=1,$$

which is closely related to the addition formula for the spherical harmonics (cf. [6, Vol. II, p. 244, (2)]).

Next we derive a compact formula for the reproducing kernel function $\mathbf{K}_n^{(\mu)}(\cdot,\cdot)$.

Theorem 3.3. *For W_μ on B^d, $\mu>0$,*

$$\begin{aligned}\mathbf{K}_n^{(\mu)}(\mathbf{x},\mathbf{y}) &= \int_0^\pi \Big[C_n^{(\mu+\frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi)\\ &\qquad + C_{n-1}^{(\mu+\frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi)\Big]\\ &\qquad\times(\sin\psi)^{2\mu-1}d\psi\Big/\int_0^\pi(\sin\psi)^{2\mu-1}d\psi\\ &= \frac{2\Gamma(\mu+\frac{d+2}{2})\Gamma(n+2\mu+d)}{\Gamma(2\mu+d+1)\Gamma(n+\mu+\frac{d}{2})}\\ &\qquad\times\int_0^\pi P_n^{(\mu+\frac{d}{2},\mu+\frac{d}{2}-1)}(\mathbf{x}\cdot\mathbf{y}+\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi)\\ &\qquad\times(\sin\psi)^{2\mu-1}d\psi\Big/\int_0^\pi(\sin\psi)^{2\mu-1}d\psi, \qquad \mathbf{x},\mathbf{y}\in B^d,\end{aligned}\tag{3.7}$$

and, for $\mu = 0$,

(3.8)

$$\begin{aligned}\mathbf{K}_n^{(0)}(\mathbf{x},\mathbf{y}) = \frac{1}{2}\Big[C_n^{(\frac{d+1}{2})}&(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \\ &+ C_{n-1}^{(\frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})\Big] \\ + \frac{1}{2}\Big[C_n^{(\frac{d+1}{2})}&(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \\ &+ C_{n-1}^{(\frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})\Big] \\ = \frac{\Gamma(\frac{d+2}{2})\Gamma(n+d)}{\Gamma(d+1)\Gamma(n+\frac{d}{2})}\Big[&P_n^{(\frac{d}{2},\frac{d}{2}-1)}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \\ &+ P_n^{(\frac{d}{2},\frac{d}{2}-1)}(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})\Big], \qquad \mathbf{x},\mathbf{y}\in B^d.\end{aligned}$$

Proof. According to [13, p. 83, (4.7.29)] we have

$$\frac{k+\lambda}{\lambda}C_k^{(\lambda)}(x) = C_k^{(\lambda+1)}(x) - C_{k-2}^{(\lambda+1)}(x), \qquad k \ge 0,$$

where $C_{-2}^{(\lambda+1)} = C_{-1}^{(\lambda+1)} = 0$, from which it follows readily that

$$\sum_{k=0}^{n}\frac{k+\lambda}{\lambda}C_k^{(\lambda)}(x) = C_n^{(\lambda+1)}(x) + C_{n-1}^{(\lambda+1)}(x).$$

The first equal sign in (3.7) is the consequence of (3.1) and this identity. From [1, p. 782, (22.7.19)] the Jacobi polynomials $P_n^{(\alpha,\beta)}$ satisfies

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)} + (n+\alpha)P_{n-1}^{(\alpha,\beta)}(x).$$

Choosing $\alpha = \beta = \lambda - 1/2$ in the formula and taking care of the normalization constants in the Jacobi and the ultraspherical polynomials (see (2.15)) we have

$$C_n^{(\lambda)}(x) + C_{n-1}^{(\lambda)}(x) = \frac{2\Gamma(\lambda+\frac{1}{2})\Gamma(n+2\lambda-1)}{\Gamma(2\lambda)\Gamma(n+\lambda-\frac{1}{2})}P_n^{(\lambda-\frac{1}{2},\lambda-\frac{3}{2})}(x), \tag{3.9}$$

from which the second equal sign in (3.7) follows readily. The proof of (3.8) is similar. □

4. Positivity and summability

In this section and the next one we discuss the summability of Cesáro means of the Fourier orthogonal series. First we recall the definition of Cesàro summability. The sequence $\{s_n\}$ is summable by Cesàro's method of order δ, (C,δ), to s if

$$\frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k}s_k$$

converges to s as $n\to\infty$. If for each $n\in\mathbb{N}_0$ s_n is the n-th partial sum of the series $\sum_{k=0}^{\infty}c_k$, the Cesàro means can be rewritten as

$$\frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta}{n-k}c_k.$$

For the basic properties of Cesàro summability see [17, Chap. III].

We denote by $S^\delta_{n,d}(W_\mu; f)$ the (C,δ) means of the Fourier orthogonal series with respect to W_μ. For later use, we also denote by $\mathbf{K}^\delta_{n,d}(W_\mu;\cdot,\cdot)$ the (C,δ) means of the orthogonal polynomial series; *i.e.*,

$$\begin{aligned}\mathbf{K}^\delta_{n,d}(W_\mu;\mathbf{x},\mathbf{y}) &= \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} \left[\mathbb{P}^{(\mu)}_k(\mathbf{x})\right]^T \mathbb{P}^{(\mu)}_k(\mathbf{y}) \\ &= \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta-1}{n-k} \mathbf{K}^{(\mu)}_{k,d}(\mathbf{x},\mathbf{y}),\end{aligned} \tag{4.1}$$

so that, by the formula (1.5), we can write

$$S^\delta_{n,d}(W_\mu; f) = \int_{B^d} f(\mathbf{y})\mathbf{K}^\delta_{n,d}(W_\mu;\mathbf{x},\mathbf{y})W_\mu(\mathbf{y})d\mathbf{y}. \tag{4.2}$$

Moreover, for $d=1$, we write $K^{(\mu)}_n$ and K^δ_n for $\mathbf{K}^{(\mu)}_{n,1}$ and $\mathbf{K}^\delta_{n,1}(W_\mu)$, respectively.

Our first result concerns the nonnegative summability method. For the ultraspherical polynomials, an inequality due to Kogbetliantz (see [2, p. 71]) states that

$$\sum_{k=0}^n \binom{n-k+2\lambda+1}{2\lambda+1} \frac{k+\lambda}{\lambda} C^\lambda_k(x) \geq 0, \tag{4.3}$$

which implies that the $(C, 2\lambda+1)$ means of the orthogonal expansion with respect to $W_{\lambda,1}$ is nonnegative. For $d \geq 1$, we have

Theorem 4.1. *For $\mu \geq 0$, the $(C, 2\mu+d)$ Cesàro means of the Fourier orthogonal series with respect to W_μ define a positive linear operator. Consequently, for $f \in C(B^d)$, the $(C, 2\mu+d)$ Cesàro means converge uniformly to f on B^d.*

Proof. The positivity of the $(C, 2\mu+d)$ means states that, by (4.1) and (4.2),

$$\sum_{k=0}^n \binom{n-k+2\mu+d}{n-k} \mathbb{P}^{(\mu)T}_k(\mathbf{x})\mathbb{P}^{(\mu)}_k(\mathbf{y}) \geq 0,$$

which follows readily, by formulae (3.1) and (3.2), from inequality (4.3) of Kogbetliantz. □

By orthogonality, the positivity implies, in particular, that the $(C, 2\mu+d)$ means are uniformly bounded. Moreover, since $S_{n,d}f$ preserves polynomials of degree less than n, it is easily seen that $S^\delta_{n,d}(W_\mu; P)$ converges uniformly to P for any polynomial P. Therefore, the uniform convergence of the $(C, 2\mu+d)$ means for the continuous functions follows readily from the density of polynomials in $C(B^d)$. □

We note that for $\mu = 0$ the theorem states that the (C, d) means are positive; in particular, for $d = 1$, the corresponding weight function becomes the Chebyshev weight of the first kind and the theorem, under the standard transformation $x \mapsto \cos\theta$, goes back to the famous Fejér theorem on the positivity of the $(C, 1)$ means of the Fourier series. Another extension of Fejér's theorem to several variables has been given in [3] for $\ell-1$ Fourier partial sums, which is equivalent to the expansion in the product Chebyshev polynomials on $[-1,1]^d$.

The order $2\mu+d$ in the theorem is sharp for the positivity. For convergence, however, the positivity is not necessary, as we shall see in the next section. In the rest of this section, we mention other results on the positivity of the polynomial sums. An inequality of Askey and Gasper for the ultraspherical polynomials states (see [2, p. 74, (8.12)]) that

$$\sum_{k=0}^{n} \binom{n-k+a}{n-k}\binom{k+a}{k}\frac{C_k^\lambda(x)}{C_k^\lambda(1)} \ge 0, \qquad 3 \le a \le 2\lambda+1, \tag{4.4}$$

for $-1 \le x \le 1$. Let us denote by the boldface letter $\mathbf{e}$ an element on the boundary of B^d; *i.e.*, $|\mathbf{e}| = 1$. It follows readily from Theorem 3.1 that

$$\left[\mathbb{P}_n^{(\mu)}(\mathbf{e})\right]^T \mathbb{P}_n^{(\mu)}(\mathbf{e}) = \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} C_n^{(\mu+\frac{d-1}{2})}(1).$$

Hence, we can rewrite formula (3.1) in Theorem 3.1 as follows:

$$\begin{aligned}\frac{\left[\mathbb{P}_n^{(\mu)}(\mathbf{x})\right]^T \mathbb{P}_n^{(\mu)}(\mathbf{y})}{\left[\mathbb{P}_n^{(\mu)}(\mathbf{e})\right]^T \mathbb{P}_n^{(\mu)}(\mathbf{e})} =& \frac{1}{C_n^{(\mu+\frac{d-1}{2})}(1)} \int_0^\pi C_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\cos\psi) \\ & \times (\sin\psi)^{2\mu-1} d\psi \Big/ \int_0^\pi (\sin\psi)^{2\mu-1} d\psi, \qquad \mathbf{x},\mathbf{y} \in B^d.\end{aligned}$$

From this formula and an analogy formula for the case $\mu = 0$, we obtain readily that

Theorem 4.2. *For $3 \le a \le 2\mu + d$,*

$$\sum_{k=0}^{n} \binom{n-k+a}{n-k}\binom{k+a}{k}\frac{[\mathbb{P}_k^{(\mu)}]^T(\mathbf{x})\mathbb{P}_k^{(\mu)}(\mathbf{y})}{\mathbb{P}_k^{(\mu)T}(\mathbf{e})\mathbb{P}_k^{(\mu)}(\mathbf{e})} \ge 0, \qquad \mathbf{x},\mathbf{y} \in B^d.$$

We note, however, that the positivity stated in Theorem 4.1 is not a special case of the inequality in Theorem 4.2. Actually, the inequality of Kogbetliantz is not a special case of (4.4), but a special case of the extension of (4.4) for the Jacobi polynomials [8]; the link can be seen from (3.9).

5. Summability of Fourier orthogonal series

The formulae we derived in Theorem 3.1 suggests that the behavior of the polynomial $\left[\mathbb{P}^{(\mu)}(\mathbf{x})\right]^T \mathbb{P}^{(\mu)}(\mathbf{y})$ resembles a polynomial of one variable. This is especially true when one of the variables is on the boundary of the ball, which suggests the following interesting observation.

Lemma 5.1. *For $\mu \ge 0$,*

$$\mathbf{K}_{n,d}^{(\mu)}(\mathbf{x},\mathbf{e}) = K_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{e}, 1), \qquad \mathbf{x} \in B^d, \quad |\mathbf{e}| = 1. \tag{5.1}$$

Proof. From (3.7) and (3.8) it follows readily that

$$\mathbf{K}_{n,d}^{(\mu)}(\mathbf{x},\mathbf{e}) = \frac{2\Gamma(\mu+\frac{d+2}{2})\Gamma(n+2\mu+d)}{\Gamma(2\mu+d+1)\Gamma(n+\mu+\frac{d}{2})} P_n^{(\mu+\frac{d}{2},\mu+\frac{d-2}{2})}(\mathbf{x}\cdot\mathbf{e}), \qquad \mu \ge 0.$$

On the other hand, from [13, p. 71, (4.5.3)] we have that

$$\tilde{K}_n^{(\alpha,\beta)}(x,1) = \frac{1}{2^{\alpha+\beta+1}} \frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)} P_n^{(\alpha+1,\beta)}(x),$$

where $\tilde{K}_n^{(\alpha,\beta)}$ is the reproducing kernel with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. From this formula we can derive the formula for $K_n^{(\mu)}(x,1)$, which is the reproducing kernel with respect to the normalized weight function

$W_{\mu,1}(x) = w_{\mu,1}(1-x^2)^{\mu-1/2}$. By setting $\alpha = \beta = \mu - 1/2$ and taking the value of $w_{\mu,1}$ in (2.2) into consideration, we obtain that

$$K_n^{(\mu)}(x,1) = \frac{2\Gamma(\mu+\frac{1}{2})\Gamma(n+2\mu+1)}{\Gamma(2\mu+1)\Gamma(n+\mu+\frac{1}{2})} P_n^{(\mu+\frac{1}{2},\mu-\frac{1}{2})}(x).$$

Letting $\mu \mapsto \mu + (d-1)/2$ and comparing the formulae for $K_n^{(\mu+\frac{d-1}{2})}(x,1)$ and $\mathbf{K}_{n,d}^{(\mu)}(\mathbf{x},\mathbf{e})$ concludes the proof. $\square$

Formula (5.1) allows us to reduce the summability on the boundary of B^d to that of one variable. The result is the following.

Theorem 5.2. *Let f be continuous on the closed ball B^d. The expansion of f in the Fourier orthogonal series with respect to W_μ is (C,δ) summable at the boundary of B^d, provided $\delta > \mu + \frac{d-1}{2}$. Moreover, it is not (C,δ) summable if $\delta = \mu + \frac{d-1}{2}$.*

Proof. By formula (4.2), it follows from Helly's theorem [13, p. 13] that it suffices to show that

$$\begin{aligned}\mathcal{I}_{n,d}^{(\mu)} &= \int_{B^d} \left|\mathbf{K}_{n,d}^{\delta}(W_\mu;\mathbf{e},\mathbf{y})\right| W_{\mu,d}(\mathbf{y})d\mathbf{y} \\ &= \int_{B^d} \left|\sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} \mathbf{K}_{k,d}^{(\mu)}(\mathbf{e},\mathbf{y})\right| W_{\mu,d}(\mathbf{y})d\mathbf{y}\end{aligned}$$

is bounded if and only if $\delta > \mu + \frac{d-1}{2}$. From Lemma 5.1 and using the standard transformation $\mathbf{y} = r\mathbf{y}'$, where $|\mathbf{y}'| = 1$, we have

$$\begin{aligned}\mathcal{I}_{n,d}^{(\mu)} &= \int_0^1 r^{d-1} \int_{S^{d-1}} \left|\sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} K_k^{(\mu+\frac{d-1}{2})}(\mathbf{y}\cdot\mathbf{e},1)\right| d\omega W_{\mu,d}(r)dr \\ &= \int_0^1 r^{d-1} \int_{S^{d-1}} \left|K_n^{\delta}(W_{\mu+\frac{d-1}{2},1};\mathbf{y}\cdot\mathbf{e},1)\right| d\omega W_{\mu,d}(r)dr.\end{aligned}$$

Since the inner integral can be viewed as an average of a function whose variable is an inner product over the sphere, it should be invariant under the orthogonal transformation; in fact, it is a radial function. We can apply the following general formula that holds for $g : \mathbb{R} \mapsto \mathbb{R}$;

$$(5.2) \qquad \int_{S^{d-1}} g(\mathbf{x}\cdot\mathbf{y})d\omega(\mathbf{y}) = \omega_{d-2} \int_{-1}^{1} g(s|\mathbf{x}|)(1-s^2)^{\frac{d-3}{2}} ds, \quad \mathbf{x} \in \mathbb{R}^d,$$

which can be easily verified and it appeared already in [9, p. 8]. It follows then that

$$\mathcal{I}_{n,d}^{(\mu)} = \omega_{d-2} \int_0^1 r^{d-1} \int_{-1}^{1} \left|K_n^{\delta}(W_{\mu+\frac{d-1}{2},1};rs,1)\right| (1-s^2)^{\frac{d-3}{2}} ds W_{\mu,d}(r)dr.$$

Making a change of variable $s \mapsto t/r$ and exchanging the order of integration, we obtain that

$$\mathcal{I}_{n,d}^{(\mu)} = \omega_{d-2} \int_{-1}^{1} \left|K_n^{\delta}(W_{\mu+\frac{d-1}{2},1};t,1)\right| \int_{|t|}^{1} (r^2-t^2)^{\frac{d-3}{2}}\ rW_{\mu,d}(r)dr\,dt.$$

Let $\phi : [-1,1] \mapsto \mathbb{R}$ be the function defined by

$$\phi(t) = \int_{|t|}^{1} (r^2-t^2)^{\frac{d-3}{2}}\ rW_{\mu,d}(r)dr = \frac{w_{\mu,d}}{2} \int_{t^2}^{1} (u-t^2)^{\frac{d-3}{2}} (1-u)^{\mu-\frac{1}{2}} du.$$

It is easy to see that the integral is a Beta integral over $[t^2, 1]$ (cf. [1, p. 258]); thus, it follows that

$$\phi(t) = A_\mu (1-t^2)^{\mu+\frac{d-2}{2}}, \quad \text{where} \quad A_\mu = \frac{w_{\mu,d}}{2} \int_0^1 u^{\frac{d-3}{2}} (1-u)^{\mu-\frac{1}{2}} du.$$

Hence, we obtain the formula

$$\mathcal{I}_{n,d}^{(\mu)} = \omega_{d-2} A_\mu \int_{-1}^1 \left| K_n^\delta (W_{\mu+\frac{d-1}{2},1}; t, 1) \right| (1-t^2)^{\mu+\frac{d-2}{2}} dt = \mathcal{I}_{n,1}^{(\mu+\frac{d-1}{2})},$$

where in the last equality we determine the constant by setting $n = 0$. Therefore, the (C, δ) summability of the orthogonal series with respect to $W_{\mu,d}$, on the boundary of B^d, is equivalent to that of the (C, δ) summability of the ultraspherical polynomial series with index $\mu + \frac{d-1}{2}$ at the end point $x = 1$. The desired result now follows from Theorem 9.1.3 in [13, p. 246], where the result is stated for the Jacobi series and a shift of 1/2 on the index is necessary for the ultraspherical series. $\square$

For $d = 1$ the condition $\delta > \mu$ turns out to be sharp for the uniform convergence of $S_n^\delta(W_{\mu,1}; f)$ on $[-1, 1]$, which, in fact, can be derived from the convergence at $x = 1$ by using a convolution structure enjoyed by the ultraspherical polynomials. The convolution structure follows from the product formula (2.13), which holds actually for the Jacobi series ([7], and see [12] for the addition formula for Jacobi polynomials). For $d \geq 1$, formula (3.1), or (3.2), is not a product formula for two individual polynomials, but holds for a sum of the product of polynomials; it's not clear whether it will yield an analogous convolution structure for the Fourier orthogonal series on B^d. Nevertheless, the formula is enough for proving the uniform convergence on B^d.

Theorem 5.3. *Let f be continuous on the closed ball B^d. The expansion of f in the Fourier orthogonal series with respect to W_μ is uniformly (C, δ) summable on B^d if, and only if, $\delta > \mu + \frac{d-1}{2}$.*

Proof. The necessary part has been proved in the previous theorem. From (4.2) and the Helly's theorem, it suffices to prove that

$$\mathcal{I}_{n,d}^{(\mu)}(\mathbf{x}) = \int_{B^d} \left| \mathbf{K}_{n,d}^\delta (W_{\mu,d}; \mathbf{x}, \mathbf{y}) \right| W_{\mu,d}(\mathbf{y}) d\mathbf{y}$$

is uniformly bounded over B^d if $\delta > \mu + (d-1)/2$. We consider the case $\mu = 0$ first. By (3.2) and (2.10) we have from (4.1) that

$$\begin{aligned} \mathbf{K}_{n,d}^\delta (W_{0,d}; \mathbf{x}, \mathbf{y}) =& \frac{1}{2} K_n^\delta (W_{\frac{d-1}{2},1}; 1, \mathbf{x} \cdot \mathbf{y} + \sqrt{1-|\mathbf{x}|^2} \sqrt{1-|\mathbf{y}|^2}) \\ &+ \frac{1}{2} K_n^\delta (W_{\frac{d-1}{2},1}; 1, \mathbf{x} \cdot \mathbf{y} - \sqrt{1-|\mathbf{x}|^2} \sqrt{1-|\mathbf{y}|^2}). \end{aligned}$$

Using the standard change of variables $\mathbf{y} = r\mathbf{y}'$, $\mathbf{y}' \in S^{d-1}$ and formula (5.2), we obtain

$$
\begin{aligned}
&\int_{B^d} \Big| K_n^\delta(W_{\frac{d-1}{2},1}; 1, \mathbf{x}\cdot\mathbf{y} \pm \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \Big| W_{0,d}(\mathbf{y}) d\mathbf{y} \\
&= \int_0^1 r^{d-1} \int_{S^{d-1}} \Big| K_n^\delta(W_{\frac{d-1}{2},1}; 1, r\mathbf{x}\cdot\mathbf{y}' \pm \sqrt{1-|\mathbf{x}|^2}\sqrt{1-r^2}) \Big| d\omega W_{0,d}(r) dr \\
&= \omega_{d-2} \int_0^1 \int_{-1}^1 r^{d-1} \Big| K_n^\delta(W_{\frac{d-1}{2},1}; 1, r|\mathbf{x}|s \\
&\qquad\qquad \pm \sqrt{1-|\mathbf{x}|^2}\sqrt{1-r^2}) \Big| (1-s^2)^{\frac{d-3}{2}} W_{0,d}(r) ds dr \\
&= \frac{a_d^{(0)}}{2} \int_0^1 \int_{-1}^1 \Big| K_n^\delta(W_{\frac{d-1}{2},1}; 1, \pm u\sqrt{1-|\mathbf{x}|^2} \\
&\qquad\qquad + |\mathbf{x}|\sqrt{1-u^2}\, s) \Big| (1-s^2)^{\frac{d-3}{2}} (1-u^2)^{\frac{d-2}{2}} ds du,
\end{aligned}
$$

where $a_d^{(0)} = \omega_{d-2} w_{0,d}$ and the last equality follows from a change of variable $r \mapsto \sqrt{1-u^2}$. Put these formulae together and it follows readily that

$$
\begin{aligned}
\mathcal{I}_{n,d}^{(0)}(\mathbf{x}) \le \frac{a_d^{(0)}}{2} \int_{-1}^1 \int_{-1}^1 \Big| K_n^\delta(W_{\frac{d-1}{2},1}; 1, uv + \sqrt{1-v^2}\sqrt{1-u^2}\, s) \Big| \\
\times (1-s^2)^{\frac{d-3}{2}} (1-u^2)^{\frac{d-2}{2}} ds du := \mathcal{J}_{n,d}^{(0)}(v),
\end{aligned} \tag{5.3}
$$

where $v = \sqrt{1-|\mathbf{x}|^2}$. From this formula, the proof will be essentially reduced to that of ultraspherical polynomials in one variable. Before we proceed, we first reduce $\mathcal{I}_{n,d}^{(\mu)}$, $\mu > 0$, to a similar form. By (3.1) and (2.10) we have from (4.1) that for $\mu > 0$,

$$
\begin{aligned}
&\mathbf{K}_{n,d}^\delta(W_{\mu,d}; \mathbf{x}, \mathbf{y}) \\
&\quad = c_\mu \int_{-1}^1 K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, \mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t)(1-t^2)^{\mu-1} dt.
\end{aligned}
$$

Therefore, changing variables $\mathbf{y} = r\mathbf{y}'$ and using (5.2) as in the case of $\mu = 0$, we follow the procedure that leads to (5.3) to conclude that

$$
\begin{aligned}
\mathcal{I}_{n,d}^{(\mu)}(\mathbf{x}) \le a_d^{(\mu)} \int_0^1 \int_{-1}^1 \int_{-1}^1 \Big| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, uvt + \sqrt{1-u^2}\sqrt{1-v^2}\, s) \Big| (1-t^2)^{\mu-1} dt \\
\times (1-s^2)^{\frac{d-3}{2}} u^{2\mu} (1-u^2)^{\frac{d-2}{2}} ds du := \mathcal{J}_{n,d}^{(\mu)}(v), \qquad v = \sqrt{1-|\mathbf{x}|^2},
\end{aligned}
$$

where $a_d^{(\mu)} = \omega_{d-2} w_{\mu,d}$ and the inequality results from moving the absolute value inside the inner most integral. Changing variable $t \mapsto p/u$ in the inner integral and exchanging the order of integrals with respect to du and dp, we obtain

$$\begin{aligned}\mathcal{J}_{n,d}^{(\mu)}(v) =& a_d^{(\mu)} \int_0^1 \int_{-u}^{u} \int_{-1}^{1} \left| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, vp + \sqrt{1-u^2}\sqrt{1-v^2}\, s) \right| \\ & \times (1-s^2)^{\frac{d-3}{2}} ds \Big(1 - \frac{p^2}{u^2}\Big)^{\mu-1} \frac{dp}{u} u^{2\mu} (1-u^2)^{\frac{d-2}{2}} du \\ =& a_d^{(\mu)} \int_{-1}^1 \int_{|p|}^{1} \int_{-1}^{1} \left| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, vp + \sqrt{1-u^2}\sqrt{1-v^2}\, s) \right| \\ & \times (1-s^2)^{\frac{d-3}{2}} ds (u^2-p^2)^{\mu-1} u (1-u^2)^{\frac{d-2}{2}} du dp;\end{aligned}$$

changing the variable again with $s \mapsto q/\sqrt{1-u^2}$ and exchanging the order of integrals with respect to du and dq, we obtain

$$\begin{aligned}\mathcal{J}_{n,d}^{(\mu)}(v) =& a_d^{(\mu)} \int_{-1}^1 \int_{|p|}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \left| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, vp + \sqrt{1-v^2}\, q) \right| \\ & \times \Big(1 - \frac{q^2}{1-u^2}\Big)^{\frac{d-3}{2}} \frac{dq}{\sqrt{1-u^2}} (u^2-p^2)^{\mu-1} u (1-u^2)^{\frac{d-2}{2}} du dp \\ =& a_d^{(\mu)} \int_{-1}^1 \int_{-\sqrt{1-p^2}}^{\sqrt{1-p^2}} \left| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, vp + \sqrt{1-v^2}\, q) \right| \\ & \times \Big[\int_{|p|}^{\sqrt{1-q^2}} (1-q^2-u^2)^{\frac{d-3}{2}} (u^2-p^2)^{\mu-1} u du \Big] dq dp.\end{aligned}$$

Upon changing variable $u^2 \mapsto x$, it is easy to see that the integral inside the square bracket is a Beta integral; it follows that

$$\int_{|p|}^{\sqrt{1-q^2}} (1-q^2-u^2)^{\frac{d-3}{2}} (u^2-p^2)^{\mu-1} u du = b_d^{(\mu)} (1-q^2-p^2)^{\mu+\frac{d-3}{2}},$$

where

$$b_d^{(\mu)} = \frac{1}{2} \int_0^1 (1-t)^{\frac{d-3}{2}} t^{\mu-1} dt = \frac{\Gamma(\frac{d-1}{2})\Gamma(\mu)}{2\Gamma(\mu+\frac{d-1}{2})}.$$

Therefore, changing variable $q \mapsto s\sqrt{1-p^2}$ in the last expression of $\mathcal{J}_{n,d}^{(\mu)}$ and using the above identity, we finally end up with

$$\begin{aligned}\mathcal{J}_{n,d}^{(\mu)}(v) = a_d^{(\mu)} b_d^{(\mu)} \int_{-1}^1 \int_{-1}^1 & \left| K_n^\delta(W_{\mu+\frac{d-1}{2},1}; 1, vp + \sqrt{1-v^2}\sqrt{1-p^2}\, s) \right| \\ & \times (1-s^2)^{\mu+\frac{d-3}{2}} (1-p^2)^{\mu+\frac{d-2}{2}} ds dp. \end{aligned} \tag{5.4}$$

Since the formula for $b_d^{(\mu)}$ implies that $b_d^{(0)} = 1/2$, we see that (5.3) corresponds to (5.4) with $\mu = 0$. We now use (5.4) to prove that $\mathcal{J}_{n,d}^{(\mu)}$ is uniformly bounded whenever $\delta > \mu + (d-1)/2$, which can be proved as in the summability for the ultraspherical series of one variable. Without introducing the convolution structure explicitly, we give a complete proof here. We start with the formula

$$\int_{-1}^1 g(uv + \sqrt{1-v^2}\sqrt{1-u^2}\, s)(1-s^2)^\lambda ds = \int_{-1}^1 g(z) D_\lambda(u,v,z)(1-z^2)^{\lambda+\frac{1}{2}} dz, \tag{5.5}$$

which follows from a change of variable, with D_λ defined by

$$D_\lambda(u,v,z) = \frac{(1-u^2-v^2-z^2+2uvz)^\lambda}{[(1-u^2)(1-v^2)(1-z^2)]^{\lambda+\frac{1}{2}}}$$

whenever $1-u^2-v^2-z^2+2uvz \ge 0$ and $D_\lambda(u,v,z)=0$, otherwise. In particular, $D_\lambda(u,v,z)$ is nonnegative and it is symmetric with respect to its three variables. Moreover, setting $g(z)=1$ yields

$$\int_{-1}^{1} D_\lambda(u,v,z)(1-z^2)^{\lambda+\frac{1}{2}}dz = \int_{-1}^{1}(1-s^2)^\lambda ds = w^{-1}_{\lambda+\frac{1}{2},1}. \tag{5.6}$$

Using formula (5.5) with $\lambda = \mu + (d-3)/2$ we obtain from (5.4) that

$$\begin{aligned}\mathcal{J}^{(\mu)}_{n,d}(v) =& a_d^{(\mu)} b_d^{(\mu)} \int_{-1}^{1}\int_{-1}^{1} \Big|K_n^\delta(W_{\mu+\frac{d-1}{2},1};1,z)\Big| D_{\mu+\frac{d-3}{2}}(u,v,z)(1-z^2)^{\mu+\frac{d-2}{2}}dz \\ & \times (1-u^2)^{\mu+\frac{d-2}{2}}du \\ =& a_d^{(\mu)} b_d^{(\mu)} \int_{-1}^{1} \Big|K_n^\delta(W_{\mu+\frac{d-1}{2},1};1,z)\Big| \Big(\int_{-1}^{1} D_{\mu+\frac{d-3}{2}}(u,v,z)(1-u^2)^{\mu+\frac{d-2}{2}}du\Big) \\ & \times (1-z^2)^{\mu+\frac{d-2}{2}}dz.\end{aligned}$$

By the symmetry of D_λ and (5.6) we have that the inner integral is equal to $[w_{\mu+\frac{d-2}{2},1}]^{-1}$ and we conclude that

$$\mathcal{J}_{n,d}(v) = \int_{-1}^{1}\Big|K_n^\delta(W_{\mu+\frac{d-1}{2},1};1,z)\Big| W_{\frac{d-1}{2},1}(z)dz,$$

where the constant is determined by the fact that

$$a_d^{(\mu)} b_d^{(\mu)} = w_{\mu+\frac{d-2}{2},1} w_{\mu+\frac{d-1}{2},1}$$

which can be easily verified using (2.2) and $\omega_{d-2} = 2\pi^{(d-1)/2}/\Gamma((d-1)/2)$. Therefore, the conclusion of the theorem follows from that of ultraspherical polynomials ([13, p. 246]); it also follows from Theorem 5.2 in view of Lemma 5.1. □

For primitive results on Cesáro summability, usually under the assumption that 2μ is an integer, we refer to [6, Vol. II, Section 12.7]. In the early studies, functions are usually expanded in the series of orthogonal polynomials $V_\alpha^{n,(\mu)}$ in (2.8), called Appell series, with the help of a second family of polynomials that is biorthogonal in connection with $V_\alpha^{n,(\mu)}$. As we mentioned before that $V_\alpha^{n,(\mu)}$ forms a basis for the space $\mathcal{V}_n^d$ spanned by $P_\alpha^{n,(\mu)}$, $|\alpha|_1 = n$; it follows from the interpretation in (1.3) that the partial sums of the Appell series should be the same as the $S_{n,d}(W_\mu, f)$. Beyond the results recorded in [6], we are not aware of any other result in this direction.

The theorem gives a complete answer to the question of uniform summability of the Fourier orthogonal series with respect to W_μ. Surprisingly, the analog result does not seem to be known for any other family of classical weight functions. Indeed, even for the product Jacobi weight functions such as $\prod_{i=1}^d (1-x_i^2)^{\mu-1/2}$ on $[-1,1]^d$, for which the orthogonal polynomials are just products of ultraspherical polynomials, the (C,δ) summability of the Fourier orthogonal series has not been studied for $\mu > 0$, while the case $\mu = 0$ reduces to the ℓ-1 summability of multiple Fourier series (cf. [3]).

To conclude this paper we mention one interesting observation for the summability of the Fourier orthogonal expansion of a radial function. For a radial function f we write $f(\mathbf{x}) = f_0(|\mathbf{x}|)$, where $f_0 : \mathbb{R}_+ \mapsto \mathbb{R}$. Because of the radial symmetry of the weight function W_μ, the Fourier orthogonal expansion of a radial function should be radial as well. This is indeed the case as can be easily seen using (5.2). Hence, we can denote by $s_{n,d}(W_\mu; f_0) : \mathbb{R}_+ \to \mathbb{R}$ the partial sum $S_{n,d}(W_{\mu,d}; f, \mathbf{x})$, $r = |\mathbf{x}|$, whenever f is a radial function.

Theorem 5.4. *Let f be a radial function. Let $\tilde{f}_0$ be defined by $\tilde{f}_0(r) = f_0(\sqrt{1-r^2})$. Then for $2\mu \in \mathbb{N}_0$,*

$$s_{n,d}(W_\mu; f_0, r) = s_{n,2\mu+1}(W_{\mu+\frac{d-1}{2}}; \tilde{f}_0, \sqrt{1-r^2}). \tag{5.7}$$

Proof. We give only an outline of the proof for $\mu > 0$; the case $\mu = 0$ is similar. We write the Fourier coefficients $\mathbf{a}_n(f)$ (see (1.3)) as $\mathbf{a}_{n,d}^{(\mu)}(f)$. By (3.1), making the change of variables $\mathbf{y} = r\mathbf{y}'$ and using (5.2) we obtain that

$$\begin{aligned}
&\left[\mathbf{a}_{n,d}^{(\mu)}(f)\right]^T \mathbb{P}_{n,d}^{(\mu)}(\mathbf{x}) = c_\mu \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} \\
&\quad \times \int_{B^d} f(\mathbf{y}) \int_{-1}^{1} C_n^{(\mu+\frac{d-1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t)(1-t^2)^{\mu-1} dt W_\mu(\mathbf{y}) d\mathbf{y} \\
&\quad = c_\mu w_\mu \frac{n+\mu+\frac{d-1}{2}}{\mu+\frac{d-1}{2}} \omega_{d-2} \int_0^1 f_0(r) \int_{-1}^{1}\int_{-1}^{1} C_n^{(\mu+\frac{d-1}{2})}(r|\mathbf{x}|s + \sqrt{1-r^2}\sqrt{1-|\mathbf{x}|^2}\, t) \\
&\qquad \times (1-t^2)^{\mu-1} dt (1-s^2)^{\frac{d-3}{2}} ds\, r^{d-1}(1-r^2)^{\mu-\frac{1}{2}} dr.
\end{aligned}$$

The last expression is radial in $\mathbf{x}$ and it depends on f_0. We denote it by $\left[\mathbf{a}_{n,d}^{(\mu)}(f_0)\right]^T \mathbb{P}_{n,d}^{(\mu)}(r)$; the change of notation is consistent with the notation of $s_{n,d}$. Changing the variable $r \mapsto \sqrt{1-u^2}$, from the fact that f_0 becomes $\tilde{f}_0$ and that $r^{d-1}(1-r^2)^{\mu-\frac{1}{2}} dr$ becomes $(1-u^2)^{\frac{d-2}{2}} u^{2\mu} du$, which is symmetric with respect to μ and $(d-1)/2$, it follows that

$$\left[\mathbf{a}_{n,d}^{(\mu)}(f_0)\right]^T \mathbb{P}_{n,d}^{(\mu)}(r) = \left[\mathbf{a}_{n,2\mu+1}^{(\frac{d-1}{2})}(\tilde{f}_0)\right]^T \mathbb{P}_{n,2\mu+1}^{(\frac{d-1}{2})}(\sqrt{1-r^2}),$$

where the constant is determined by setting $n = 0$. The desired result follows readily. ☐

In particular, when $\mu = 0$, the formula (5.7) states that

$$s_{n,d}(W_0; f_0, r) = s_{n,1}(W_{\frac{d-1}{2}}; \tilde{f}_0, \sqrt{1-r^2}).$$

Thus, for a radial function f on B^d, its expansion in the Fourier orthogonal series with respect to $W_{0,d}$ can be reduced to the expansion of $\tilde{f}_0$ in the ultraspherical series.

Finally let us mention that the compact formula also makes it possible to study the asymptotics of $[\mathbb{P}_n(\mathbf{x})]^T \mathbb{P}_n(\mathbf{x})$, from which the asymptotics of the Christoffel function $[\mathbf{K}_n(\mathbf{x},\mathbf{x})]^{-1}$ in [5] follows; the result will be presented in another place.

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